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TIRT STUDY NO. 9: STOKES VECTOR FORMALISM FOR THE SOURCE RADIANCE OF AN OPAQUE BODY

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FOREWORD

This paper was prepared for the Advanced Vehicle Systems Technology Office, Defense Advanced Research Projects Agency, under DARPA Project Assignment A-101B. Task A-101--Project TIRT (Topics in Infrared Technology)--is an infrared technology base program.

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ABSTRACT

A simple and rigorous formalism is presented for describing the incoherent radiative properties of absorption, emission, and scattering of an opaque body in local thermodynamic equilibrium; polarization, inelastic scattering, and applied magnetic field effects are treated in full. The radiative behavior of such a body is shown to be completely characterized by the local bispectral bidirectional reflectivity matrix. Expressions for the emitted and reflected Stokes vectors of the source radiation are given in terms of this matrix. Use is made of the most general forms of the reciprocal relations and Kirchhoff's law; derivations for these are also provided.

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EXECUTIVE SUMMARY

This paper presents a formalism for completely characterizing the incoherent radiative properties of an opaque body in local thermodynamic equilibrium. Polarization (described by means of the Stokes vector), inelastic scattering, and applied magnetic field effects (covering both "reciprocal" and "nonreciprocal" media) are treated in full--to our knowledge, for the first time.

Three local material properties are separately introduced to describe the distinct processes of absorption, emission, and scattering; these are, respectively, the absorptivity vector, the emissivity vector, and the bispectral bidirectional reflectivity matrix. We derive rigorous expressions for the emitted and reflected components of the body's source radiance based on these properties. In obtaining our results use is made of the most general forms of the reciprocal relations and Kirchhoff's law; we provide derivations for these in Appendix A.

Perhaps our most useful result is a demonstration that the incoherent radiative properties of an opaque body in local thermodynamic equilibrium are, in fact, completely characterized by the local bispectral bidirectional reflectivity matrix alone, and that the absorptivity and emissivity vectors are mere derivative properties which uniquely follow from it. This is clearly an important result for numerical modelers who have been erroneously assuming these properties to be independent. It is also important to note that a knowledge of the absorptivity and emissivity vectors alone, on the other hand, does not constitute a complete characterization of an opaque body's radiative behavior.

Our results are also important for experimentalists. For instance, we show that a complete knowledge of the local bispectral bidirectional reflectivity, which is a 4×4 matrix, in general requires the separate measurement of each of the 16 matrix elements, each of which may be a function of as many as 13 continuous variables (viz., spatial position, incident and reflected directions and wavelengths, applied magnetic field, and temperature).

Section 6 contains a discussion of a number of implications of our results. Relevant formulas and discussion for the special (and perhaps most familiar) case of a "reciprocal" body which scatters radiation elastically and which is exposed to unpolarized incident radiation are also provided there. These results may be sufficient for readers with limited interests.

We also present a discussion of an interesting example of a simple yet practical "nonreciprocal" system in Appendix B.

1. INTRODUCTION

The ability to characterize the radiative properties of material media is important in many applications, including heat transfer engineering, remote sensing, and target signature prediction and control. A complete description must, of course, include the properties of absorption, emission, and scattering, and must cover the polarization state, as well as the total intensity, of the radiation involved. A complete characterization in the most general case is quite involved.[1]

An extremely important special case of the most general problem is that of characterizing the incoherent radiative properties of an opaque body in local thermodynamic equilibrium. In this particular case, a considerable reduction in the complexity of the description is possible. Remarkably, a thorough treatise on this important case does not appear to exist.[2]

It is our ambition here to fill this void. In particular, it is our purpose to present a simple yet rigorous *formalism for completely* describing the incoherent radiative properties of an opaque body in local thermodynamic equilibrium. Polarization (described by means of the Stokes vector), inelastic scattering, and applied magnetic field effects (covering both "reciprocal" and "nonreciprocal" media) are treated in full.

Three local material properties are separately introduced to describe the distinct processes of absorption, emission, and scattering; these are, respectively, the absorptivity vector, the emissivity vector, and the bispectral bidirectional reflectivity matrix. Perhaps our most useful result is a demonstration that the incoherent radiative behavior of an opaque body in local thermodynamic equilibrium can, in fact, be completely characterized by the local bispectral bidirectional reflectivity matrix alone, and that the absorptivity and emissivity vectors are mere derivative properties which uniquely follow from it (and which, by themselves, do not in general constitute a complete characterization). We show that a complete knowledge of this 4×4 matrix for an opaque body in general requires the separate measurement of each of the 16 matrix elements, each of which may be a function of as many as 13 continuous variables (viz., spatial position, incident and reflected directions and wavelengths, applied magnetic field, and temperature). We give explicit expressions for the emitted and reflected Stokes vectors of the source radiation in terms of this matrix.

Our paper is organized as follows: In Section 2, we briefly review the Stokes vector formalism for radiation. In terms of this formalism, we introduce in Section 3 the bispectral bidirectional reflectivity matrix and the emissivity and absorptivity vectors. In Section 4, we derive the absorptivity and emissivity vectors of an opaque body from the bispectral bidirectional reflectivity matrix. In Section 5, we use the results of Section 4 to derive a simple expression for the full Stokes vector of the source radiation of an opaque body in local thermodynamic equilibrium; this expression is in terms of this reflectivity matrix alone, without reference to the emissivity vector. In Section 6, we discuss briefly some of the implications of our results. Formulas pertaining to the special (and perhaps most familiar) case of a "reciprocal" opaque body which scatters radiation elastically, and which is exposed to unpolarized incident radiation, are also provided there; of these, Eq. (12') deserves special note.

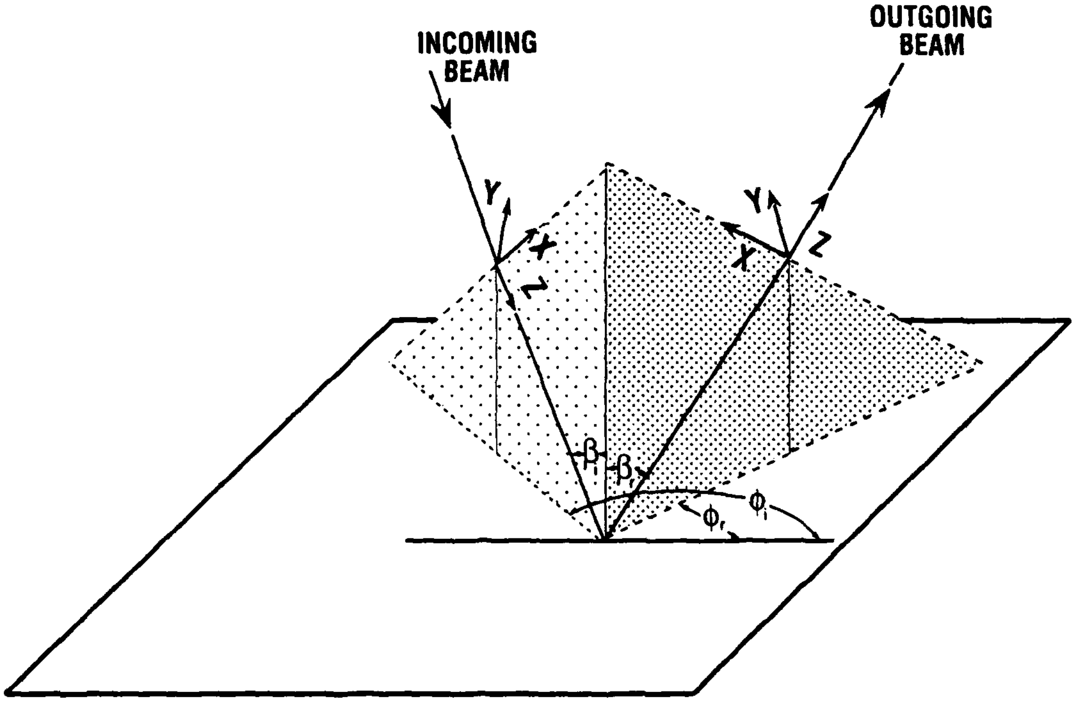
In obtaining our results we make use of the most general expressions of the reciprocal relations and Kirchhoff's law; these are derived in Appendix A. An example of a simple yet practical "nonreciprocal" system is presented and discussed in Appendix B. Finally, a careful discussion of assumptions is given in Appendix C.

2. THE STOKES VECTOR

In this section we present a brief review of the Stokes vector formalism,[3] since it is with respect to this formalism that we shall subsequently characterize the radiative properties of matter. Standard convention is followed.

Incoherent electromagnetic radiation is completely described by the local spectral directional four-component Stokes vector $\mathbb{L}(\mathbb{R}, \mathbf{\Omega}, \lambda)$; here $\mathbb{R}$ is a position in space, $\mathbf{\Omega}$ is the propagation axis (the actual direction of propagation along this axis will be clear from context), and λ is the wavelength. In this article we shall be concerned only with positions $\mathbb{R}$ on the surface of an opaque body. $\mathbf{\Omega}$ will be defined by a polar angle β and an azimuthal angle ϕ measured with respect to the local outward-directed surface normal of the body at $\mathbb{R}$, and will itself be always taken to point away from the body.

We give an operational definition of the local spectral directional Stokes vector (where familiarity with linearly and circularly polarizing filters is assumed), and, for definiteness, we express its four components L_0, L_1, L_2, L_3 with respect to the following local Cartesian coordinates (see Figure): a z-axis in the direction of propagation; an x-axis in the plane containing the z-axis and the surface normal, pointing away from the local surface; and a y-axis normal to this plane so as to give an orthogonal xyz right-handed system. L_0 is then the full spectral directional radiance of the radiation (i.e., the total power of the electromagnetic field, per unit projected area, solid angle, and wavelength); L_1 is the difference F_1-L_0, where F_1 is twice the spectral directional radiance transmitted by a perfect linear polarizer which has unity transmission for the electric field components in the $x = 0$ plane and is opaque to electric field components in the $y = 0$ plane; L_2 is the difference F_2-L_0, where F_2 is twice the spectral directional radiance transmitted by the same filter as before, but which has been rotated so that the transmission plane is the $x = y$ plane; and L_3 is the difference F_3-L_0, where F_3 is twice the spectral directional radiance transmitted by a filter which has unity transmission for right-circularly polarized radiation (electric field rotating clockwise when viewed in the direction of propagation) and is opaque to left-circularly polarized radiation.



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Coordinate Systems for the Description of Incoming (Incident) and Outgoing (Emitted and Reflected) Radiation. The Choice for the Origin for the Azimuthal Coordinates ϕ_i and ϕ_r is Arbitrary.

In terms of the Stokes vector, radiation is categorized as follows (note that from the definition of $\mathbb{L}$, it can be shown that all radiation must satisfy the inequality $L_0^2 \geq L_1^2 + L_2^2 + L_3^2$): Completely polarized radiation is characterized by a Stokes vector with $L_0^2 = L_1^2 + L_2^2 + L_3^2$; such radiation is further described as being linearly polarized if $L_3 = 0$, circularly polarized if $L_1 = L_2 = 0$, and elliptically polarized if otherwise. Completely unpolarized radiation is characterized by a Stokes vector with $L_1 = L_2 = L_3 = 0$. Radiation for which $L_1^2 + L_2^2 + L_3^2$ is neither L_0^2 nor 0 is termed partially polarized.

The primary advantage of the Stokes vector formalism over other possible alternatives is additivity for incoherent radiation: The Stokes vector of an electromagnetic field formed by spatially superimposing two incoherent fields is simply the sum of the Stokes vectors of the individual fields. Another useful feature of this formalism is the immediate physical significance of the component of the Stokes vector $\mathbb{L}$ along any vector $\mathbb{P}$ of the form $(1, P_1, P_2, P_3)$, where $P_1^2 + P_2^2 + P_3^2 = 1$: The quantity $(1/2)[\mathbb{L} \bullet \mathbb{P}]$ in fact represents the spectral directional radiance associated with those components of the full underlying electric field which, by themselves, form a completely polarized field described by a Stokes vector in the direction of $\mathbb{P}$. The Stokes vector of this component field is thus $(1/2)[\mathbb{L} \bullet \mathbb{P}]\mathbb{P}$. Note that $\mathbb{P}$ is a Stokes vector which describes a completely polarized radiation field. The photons which are associated with this field are referred to as being in the polarization state $\mathbb{P}$.

3. THE RADIATIVE PROPERTIES $\boldsymbol{\rho}$, $\boldsymbol{\varepsilon}$, AND $\boldsymbol{\alpha}$

In terms of the Stokes vector formalism, we now define three local material surface properties of a body: the 4×4 bispectral bidirectional reflectivity matrix $\boldsymbol{\rho}$ (also known as the Mueller or Stokes matrix for the reflection of incoherent radiation); the four-component emissivity vector $\boldsymbol{\varepsilon}$; and the four-component absorptivity vector $\boldsymbol{\alpha}$:

$$\lambda_r \mathbf{L}^r(\mathbf{R}, \boldsymbol{\Omega}_r, \mathbf{B}_o, \lambda_r) = \int \boldsymbol{\rho}(\mathbf{R}, \boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \bullet \lambda_i \mathbf{L}^i(\mathbf{R}, \boldsymbol{\Omega}_i, \lambda_i) \cos\beta_i d\boldsymbol{\Omega}_i d\lambda_i, \quad (1)$$

$$\mathbf{L}^e(\mathbf{R}, \boldsymbol{\Omega}_r, \mathbf{B}_o, \lambda_r, T) = \boldsymbol{\varepsilon}(\mathbf{R}, \boldsymbol{\Omega}_r, \mathbf{B}_o, \lambda_r) L_{bb}(\lambda_r, T) \quad , \quad (2)$$

$$L_0^a(\mathbf{R}, \boldsymbol{\Omega}_r, \mathbf{B}_o, \lambda_r) = \boldsymbol{\alpha}(\mathbf{R}, \boldsymbol{\Omega}_r, \mathbf{B}_o, \lambda_r) \bullet \mathbf{L}^i(\mathbf{R}, \boldsymbol{\Omega}_r, \lambda_r) \ . \quad (3)$$

In Eqs. (1)-(3), $\mathbf{L}^r$, $\mathbf{L}^i$, and $\mathbf{L}^e$ are the local spectral directional Stokes vectors associated, respectively, with the reflected, incident, and emitted radiation; $\mathbf{B}_o$ is any time-independent (or very slowly varying) externally applied magnetic field; $L_{bb}(\lambda, T)$ is the blackbody directional spectral radiance at the temperature T of the surface point $\mathbf{R}$; and L_0^a is the local directional spectral radiance absorbed from the incident radiation $\mathbf{L}^i$. The integration indicated in Eq. (1) is to be performed over the full hemisphere of incidence angles at the surface point $\mathbf{R}$, and over all incident wavelengths. The "$\bullet$" indicates a matrix multiplication between the matrix $\boldsymbol{\rho}$ and the vector $\mathbf{L}^i$ in Eq. (1) and a scalar product between the vectors $\boldsymbol{\alpha}$ and $\mathbf{L}^i$ in Eq. (3).

The first component ε_0 of the emissivity vector $\boldsymbol{\varepsilon}$ corresponds to the usual quantity referred to in the literature as the local spectral directional "emissivity" and represents the ratio of the local spectral directional radiance emitted by the body to that emitted by a blackbody (i.e., L_{bb}). Likewise, the first component α_0 of the absorptivity vector $\boldsymbol{\alpha}$ corresponds to the usual quantity referred to in the literature as the local spectral directional "absorptivity" and represents the fraction of the local spectral directional radiance absorbed by the body from an unpolarized incident beam of radiation. And, finally, the quantity $\rho_{00}/\delta(\lambda_r-\lambda_i)$ corresponds to the usual quantity referred to in the literature as the local spectral "bidirectional reflectivity" or "bidirectional distribution function" for bodies which only reflect radiation elastically [see discussion surrounding Eq. (13) below] and represents the local spectral directional radiance produced in reflection per unit local spectral directional irradiance incident on the body.

We note here that the product $(\lambda/hc)L_0(\mathbb{R}, \boldsymbol{\Omega}, \lambda)$, where h is Planck's constant and c is the speed of light, represents the local spectral directional radiance expressed in terms of photons (rather than energy) per unit time, projected area, solid angle, and wavelength. We also point out that, while they are not explicitly shown as such in Eqs. (1)-(3), $\boldsymbol{\rho}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\alpha}$ can be temperature-dependent quantities. Henceforth, for simplicity, we shall often suppress T (and also $\mathbb{R}$) in the argument list of other quantities as well.

4. EXPRESSIONS FOR $\boldsymbol{\alpha}$ AND $\boldsymbol{\varepsilon}$ IN TERMS OF $\boldsymbol{\rho}$

The local quantities $\boldsymbol{\rho}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\alpha}$ defined in the previous section are not independent for an opaque body. In fact, we now show, by keeping track of photons, and by using reciprocity (in its most general sense) and Kirchhoff's law, that $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ are uniquely determined by $\boldsymbol{\rho}$. We begin by obtaining the relationship between $\boldsymbol{\alpha}$ and $\boldsymbol{\rho}$.

The number of photons in a wavelength band $d\lambda_i$ centered on the wavelength λ_i, which are incident per unit time on a surface element dA centered on the position $\mathbb{R}$ of the surface of a body, and which are arriving from a solid angle $d\boldsymbol{\Omega}_i$ centered on the direction $\boldsymbol{\Omega}_i$, is given in terms of the incident radiance L_0^i by: $(\lambda_i/hc)L_0^i(\boldsymbol{\Omega}_i, \lambda_i)\cos\beta_i dAd\boldsymbol{\Omega}_i d\lambda_i$. If the body is opaque, by definition all of these photons must be either reflected or absorbed. The rate $d\dot{N}_a$ at which the photons are absorbed is, from Eq. (3), given by:

$$d\dot{N}_a = \sum_{j=0}^{3} \alpha_j(\boldsymbol{\Omega}_i, \mathbb{B}_o, \lambda_i)(\lambda_i/hc)L_j^i(\boldsymbol{\Omega}_i, \lambda_i)\cos\beta_i dAd\boldsymbol{\Omega}_i d\lambda_i \quad .$$

The rate at which the photons are reflected is obtained from the first component of the vector Eq. (1) (in differential form) by first multiplying both sides by $(1/hc)\cos\beta_r dAd\boldsymbol{\Omega}_r d\lambda_r$ and then integrating both sides over all reflected photon directions and wavelengths. The right-hand side of the resulting equation for the rate $d\dot{N}_r$ at which the photons are reflected is given by (where the indicated integration is over the hemisphere of reflected angles $\boldsymbol{\Omega}_r$ and all reflected wavelengths λ_r, and *not* over the hemisphere of incident angles $\boldsymbol{\Omega}_i$ or the incident wavelengths λ_i):

$$d\dot{N}_r = \sum_{j=0}^{3} \int \rho_{0j}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbb{B}_o, \lambda_r, \lambda_i)\lambda_i L_j^i(\boldsymbol{\Omega}_i, \lambda_i)\cos\beta_i d\boldsymbol{\Omega}_i d\lambda_i (1/hc)\cos\beta_r dAd\boldsymbol{\Omega}_r d\lambda_r \ .$$

Equating the rate at which the photons are incident with the sum of the rate at which they are reflected and absorbed, we have, after division through by the common factor $(\lambda_i/hc)\cos\beta_i dAd\boldsymbol{\Omega}_i d\lambda_i$:

$$L_0^i(\boldsymbol{\Omega}_i, \lambda_i) = \sum_{j=0}^{3} \left\{ \int \rho_{0j}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i)\, L_j^i(\boldsymbol{\Omega}_i, \lambda_i) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r + \alpha_j(\boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_i)\, L_j^i(\boldsymbol{\Omega}_i, \lambda_i) \right\} . \quad (4)$$

Dividing both sides of Eq. (4) by $L_0^i(\boldsymbol{\Omega}_i, \lambda_i)$, we then obtain, after simple rearrangement:

$$1 - \alpha_0(\boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_i) - \int \rho_{00}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r$$
$$= \sum_{j=1}^{3} \frac{L_j^i(\boldsymbol{\Omega}_i, \lambda_i)}{L_0^i(\boldsymbol{\Omega}_i, \lambda_i)} \left\{ \alpha_j(\boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_i) + \int \rho_{0j}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r \right\} . \quad (5)$$

Since Eq. (5) must hold for any physically realizable Stokes vector $\mathbb{L}^i$, each of the terms in $[L_j^i(\boldsymbol{\Omega}_i, \lambda_i)/L_0^i(\boldsymbol{\Omega}_i, \lambda_i)]$ must vanish individually; thus, the coefficient of each term must be identically zero and we must require:

$$\alpha_0(\boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_i) = 1 - \int \rho_{00}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r$$
$$\alpha_j(\boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_i) = - \int \rho_{0j}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r \, ; \quad j = 1,2,3 \; . \quad (6)$$

We now make use of reciprocity and Kirchhoff's law to obtain the relationship between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$. The general reciprocal relations on $\boldsymbol{\rho}$ follow from microscopic reversibility arguments (see Appendix A):

$$\lambda_i L_{bb}(\lambda_i, T) \rho_{jk}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) = (-1)^{\delta_{2j} + \delta_{2k}} \lambda_r L_{bb}(\lambda_r, T) \rho_{kj}(\boldsymbol{\Omega}_i, \boldsymbol{\Omega}_r, -\mathbf{B}_o, \lambda_i, \lambda_r) , \quad (7)$$

where j, k = 0, 1, 2, 3 and where δ_{jk} is the Kronecker delta. Kirchhoff's law, itself a reciprocity relation between the absorptivity and emissivity, follows from similar reversibility arguments (see Appendix A):

$$\alpha_j(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda_r) = (-1)^{\delta_{2j}} \varepsilon_j(\mathbf{\Omega}_r, -\mathbf{B}_o, \lambda_r) \tag{8}$$

for j = 0, 1, 2, 3.

The dependence of the terms on opposite sides of Eqs. (7) and (8) on opposite signs of the applied magnetic field $\mathbf{B}_o$ is to be noted. The expressions as written represent generalizations of the reciprocal relations and Kirchhoff's law to include "nonreciprocal" materials (i.e., media whose radiative properties are dependent on the direction of $\mathbf{B}_o$), as well as materials of the far more familiar "reciprocal" variety (i.e., media whose radiative properties are independent of the direction of $\mathbf{B}_o$). Equations (7) and (8) enable "nonreciprocal" materials to exhibit some rather interesting and unusual behavior; we discuss an example in Appendix B.

Inserting Eq. (7) into the right-hand side of Eq.(6) and inserting Eq. (8) into the left-hand side of Eq. (6), we then obtain:

$$\varepsilon_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda_r) = 1 - \int \rho_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \frac{\lambda_i L_{bb}(\lambda_i)}{\lambda_r L_{bb}(\lambda_r)} \cos\beta_i d\mathbf{\Omega}_i d\lambda_i$$

$$\varepsilon_j(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda_r) = - \int \rho_{j0}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) \frac{\lambda_i L_{bb}(\lambda_i)}{\lambda_r L_{bb}(\lambda_r)} \cos\beta_i d\mathbf{\Omega}_i d\lambda_i \; ; \; j = 1,2,3 \; . \tag{9}$$

Equations (9) and (6) demonstrate explicitly that the local spectral directional emissivity $\boldsymbol{\varepsilon}$ and absorptivity $\boldsymbol{\alpha}$ are properties of an opaque body which are derivable from the local bispectral bidirectional reflectivity $\boldsymbol{\rho}$ alone.

5. THE SOURCE RADIANCE IN TERMS OF ρ

We may use Eq. (9) to obtain a most compact and simple expression describing the combined emitted and reflected source radiation leaving a point $\mathbb{R}$ on the surface of an opaque body. The full local spectral directional Stokes vector $\mathbb{L}$ of the source radiation is simply the sum of the emitted and reflected local spectral directional Stokes vectors:

$$\mathbb{L} = \mathbb{L}^e + \mathbb{L}^r \quad . \tag{10}$$

If we substitute into Eq. (10) from Eqs. (1), (2), and (9) and define modified Stokes vectors $\tilde{\mathbb{L}}$ and $\tilde{\mathbb{L}}^i$ as

$$\tilde{\mathbb{L}} = \mathbb{L} - L_{bb}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\tilde{\mathbb{L}}^i = \mathbb{L}^i - L_{bb}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad , \tag{11}$$

where L_{bb} is the blackbody spectral directional radiance at the temperature of the surface point, we obtain finally:

$$\lambda_r \tilde{\mathbb{L}}(\mathbb{R}, \mathbf{\Omega}_r, \mathbb{B}_o, \lambda_r, T) = \int \rho\,(\mathbb{R}, \mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbb{B}_o, \lambda_r, \lambda_i) \bullet \lambda_i\, \tilde{\mathbb{L}}^i(\mathbb{R}, \mathbf{\Omega}_i, \lambda_i, T) \cos\beta_i d\mathbf{\Omega}_i d\lambda_i \, . \tag{12}$$

The similarity in form of this result to Eq. (1) is to be noted; its simplicity should make it fairly easy to remember.

6. CONCLUSIONS

In this section we discuss some of the implications of our results.

The degrees of freedom of $\boldsymbol{\rho}$. We have demonstrated that the incoherent radiative properties of an opaque body are completely characterized by the local bispectral bidirectional reflectivity matrix $\boldsymbol{\rho}$. Aside from the reciprocal relations of Eq. (7), it can be shown that the elements ρ_{jk} of this matrix are in general only constrained by a set of inequalities.[4] Hence, aside from the factor-of-two reduction in data afforded by Eq. (7), a complete characterization of the local radiative properties of an opaque body in general requires measurement of all 16 elements ρ_{jk}, each of which is in general a function of 13 variables (viz., $\mathbf{R}$, $\boldsymbol{\Omega}_r$, $\boldsymbol{\Omega}_i$, $\mathbf{B}_o$, λ_r, λ_i,T). We also note, however, that everyday surfaces usually show significant dependence on only a few of these. For instance, surfaces typically reflect radiation with virtually no change in wavelength (i.e., elastically); thus, ordinarily, an exceedingly good approximation is

$$\boldsymbol{\rho}(\mathbf{R}, \boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) = \boldsymbol{\rho}'(\mathbf{R}, \boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbf{B}_o, \lambda_r)\, \delta(\lambda_r - \lambda_i) \quad , \tag{13}$$

where δ is the Dirac delta function and $\boldsymbol{\rho}'$ is known as the local spectral bidirectional reflectivity matrix. Furthermore, $\boldsymbol{\rho}'$ itself can for many simple surfaces often be regarded to be a function of only four variables (i.e., β_r, β_i, $\phi_r - \phi_i$, and λ_r).

If, for instance, we confine our attention to the simple but commonly encountered situation of an elastically scattering, "reciprocal" opaque surface exposed to unpolarized radiation, and if we are interested only in the radiance (without regard to polarization state) of the radiation leaving the surface, the only relevant material properties are α_0, ε_0, and ρ_{00}. Some relevant local formulas for this special case are [from Eqs. (6)-(9) and (12), respectively]:

$$\alpha_0(\boldsymbol{\Omega}_r, \lambda) = 1 - \int \rho'_{00}(\boldsymbol{\Omega}_i, \boldsymbol{\Omega}_r, \lambda) \cos\beta_i d\boldsymbol{\Omega}_i \quad , \tag{6'}$$

$$\rho'_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \lambda) = \rho'_{00}(\mathbf{\Omega}_i, \mathbf{\Omega}_r, \lambda) \ , \tag{7'}$$

$$\alpha_0(\mathbf{\Omega}_r, \lambda) = \varepsilon_0(\mathbf{\Omega}_r, \lambda) \ , \tag{8'}$$

$$\varepsilon_0(\mathbf{\Omega}_r, \lambda) = 1 - \int \rho'_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \lambda)\cos\beta_i d\mathbf{\Omega}_i \ , \tag{9'}$$

$$L_0(\mathbf{\Omega}_r, \lambda, T) = L_{bb}(\lambda, T) + \int \rho'_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \lambda)\left[L_0^i(\mathbf{\Omega}_i, \lambda) - L_{bb}(\lambda, T)\right]\cos\beta_i d\mathbf{\Omega}_i \ , \tag{12'}$$

where we have been able to suppress $\mathbb{B}_o$ and subscripts on λ.

The use of $\boldsymbol{\rho}$ in predicting the results of reflection measurements. It is a trivial matter to calculate the output spectral directional radiance from any local reflection measurement, once the bispectral bidirectional reflectivity matrix is known. For example, a typical experiment might be performed as follows. A polarizing filter which selects the polarization component along $\mathbb{I} = (1, I_1, I_2, I_3)$, where we have $I_1^2 + I_2^2 + I_3^2 = 1$, is placed in the path of an incident beam with local spectral directional Stokes vector $\mathbb{L}^i$. The incident beam is directionally filtered so that it arrives only within the small solid angle $d\mathbf{\Omega}_i$ centered on the direction $\mathbf{\Omega}_i$, and it is spectrally filtered so that it contains wavelengths only within a small band $d\lambda_i$ centered on the wavelength λ_i. (The irradiance of the beam incident on the first polarizing filter is thus $L_0^i\cos\beta_i d\mathbf{\Omega}_i d\lambda_i$.) A second polarizing filter which selects the polarization component along $\mathbb{O} = (1, O_1, O_2, O_3)$, where we have $O_1^2 + O_2^2 + O_3^2 = 1$, is placed in the path of the desired output direction $\mathbf{\Omega}_r$, in front of a radiation detector. From the formulas given at the end of Section 2 and from Eq. (1), the spectral directional radiance dL_0^r at wavelength λ_r which would be measured by the detector in this experiment is given by the simple expression

$$dL_0^r = (1/4)(\mathbb{O}\cdot\boldsymbol{\rho}\cdot\mathbb{I})(\mathbb{I}\cdot\mathbb{L}^i)(\lambda_i/\lambda_r)\cos\beta_i d\mathbf{\Omega}_i d\lambda_i \ . \tag{14}$$

(Conversely, $\boldsymbol{\rho}$ itself is generally obtained by measuring this quantity for a complete set of independent choices of $\mathbb{I}$ and $\mathbb{O}$, as well as for all other variables upon which $\boldsymbol{\rho}$ depends.)

A further constraint on $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ for opaque bodies. For nonreciprocal media, Eq. (8) allows the interesting possibility that, for a given direction $\boldsymbol{\Omega}_r$, wavelength λ_r, and applied magnetic field $\mathbb{B}_o$, the absorptivity and emissivity are independent; for instance, α_0 can be unity and ε_0 zero, or vice versa (for an example, see Appendix B). For opaque bodies, however, there exists a global constraint between α_0 and ε_0 for a given applied field $\mathbb{B}_o$, since from Eqs. (9) and (6) we have

$$\varepsilon_0(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_r, T) = 1 - \int \rho_{00}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbb{B}_o, \lambda_r, \lambda_i) \frac{\lambda_i L_{bb}(\lambda_i, T)}{\lambda_r L_{bb}(\lambda_r, T)} \cos\beta_i d\boldsymbol{\Omega}_i d\lambda_i$$

$$\alpha_0(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_r, T) = 1 - \int \rho_{00}(\boldsymbol{\Omega}_i, \boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_i, \lambda_r) \cos\beta_i d\boldsymbol{\Omega}_i d\lambda_i \quad , \tag{15}$$

from which it follows that:

$$\int \lambda_r L_{bb}(\lambda_r, T) \alpha_0(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_r, T) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r = \int \lambda_r L_{bb}(\lambda_r, T)\, \varepsilon_0(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_r, T) \cos\beta_r d\boldsymbol{\Omega}_r d\lambda_r \,. \tag{16}$$

Absolute bounds on the source radiance L_0 of an opaque body. While a complete characterization of the radiative behavior of an opaque body in general requires knowledge of $\boldsymbol{\rho}$, the local source spectral directional radiance can, in fact, be bounded if only the local spectral directional emissivity ε_0 is known. As an example of the type of inequality that can be obtained, we again consider the common situation where all incident radiation is unpolarized, and the body scatters radiation elastically; then, from Eqs. (1) and (13):

$$L_0^r(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda) = \int \rho'_{00}(\boldsymbol{\Omega}_r, \boldsymbol{\Omega}_i, \mathbb{B}_o, \lambda)\, L_0^i(\boldsymbol{\Omega}_i, \lambda) \cos\beta_i d\boldsymbol{\Omega}_i \quad . \tag{17}$$

Since ρ'_{00} is necessarily a nonnegative quantity, L_0^r must satisfy:

$$L_0^{i,max}(\lambda)\int \rho'_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbf{B}_o, \lambda) \cos\beta_i d\mathbf{\Omega}_i$$

$$\geq L_0^r(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda) \geq$$

$$L_0^{i,min}(\lambda)\int \rho'_{00}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbf{B}_o, \lambda) \cos\beta_i d\mathbf{\Omega}_i \quad , \tag{18}$$

where $L_0^{i,max}(\lambda)$ and $L_0^{i,min}(\lambda)$ are, respectively, the maximum and minimum values of $L_0^i(\mathbf{\Omega}_i, \lambda)$ with respect to $\mathbf{\Omega}_i$ at each λ. Substituting from Eq. (9), the preceding inequality can be rewritten as:

$$L_0^{i,max}(\lambda)[1 - \varepsilon_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda)]$$

$$\geq L_0^r(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda) \geq$$

$$L_0^{i,min}(\lambda)[1 - \varepsilon_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda)] \quad . \tag{19}$$

Finally, using Eqs. (10) and (2), we obtain our desired result:

$$L_0^{i,max}(\lambda) + [L_{bb}(\lambda,T) - L_0^{i,max}(\lambda)]\varepsilon_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda)$$

$$\geq L_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda, T) \geq$$

$$L_0^{i,min}(\lambda) + [L_{bb}(\lambda,T) - L_0^{i,min}(\lambda)]\varepsilon_0(\mathbf{\Omega}_r, \mathbf{B}_o, \lambda) \quad . \tag{20}$$

REFERENCES AND NOTES

1. S. Chandrasekhar, *Radiative Transfer*, Dover, New York, 1960.

2. This problem has received a great deal of partial treatment elsewhere. As good examples, Ref. 5 provides an extensive discussion of the radiance properties (without regard to polarization state) of elastically scattering, reciprocal media exposed to unpolarized radiation, and Ref. 6 provides an excellent discussion of the polarization properties of such media in the more general case of exposure to polarized radiation.

3. W.A. Shurcliff, *Polarized Light*, Harvard University Press, Cambridge, Mass., 1962.

4. J.W. Hovenier, H.C. van de Hulst, and C.V.M. van der Mee, "Conditions for the Elements of the Scattering Matrix," *Astron. Astrophys.* **157**, 301 (1986).

5. R. Siegel and J.R. Howell, *Thermal Radiation Heat Transfer*, 2nd ed., McGraw-Hill, New York, 1981.

6. H.C. van de Hulst, *Multiple Light Scattering*, Vol. 1, Academic Press, New York, 1980, Chapter 3. See also p. 173 of Ref. 1.

APPENDIX A
DERIVATION OF THE GENERAL FORMS OF KIRCHHOFF'S LAW AND THE RECIPROCITY RELATIONS

In this Appendix we provide a derivation of the complete expressions of Kirchhoff's law and the reciprocity relations on $\boldsymbol{\rho}$, including polarization, inelastic scattering, and applied magnetic field effects. While we are aware of the existence in the literature of partial versions of these laws,[1 2] to our knowledge the most general forms and their derivations have not appeared previously.

Kirchhoff's law and the reciprocity relations on $\boldsymbol{\rho}$ are physical constraints on the radiative properties of matter which follow solely from the form of the fundamental microscopic equations of motion that govern the behavior of all physical systems. The form of these equations is such that the simultaneous substitutions $+t \rightarrow -t$ and $+\mathbb{B} \rightarrow -\mathbb{B}$ (where t is the time and $\mathbb{B}$ is the magnetic field) leave the equations unchanged. This invariance implies that for every "forward" solution to the equations of motion there is a "reverse" solution; viz., if a system can evolve through a sequence of instantaneous microscopic states in one order (e.g., the "forward" one), and if all the externally applied magnetic fields are first reversed, the system can also evolve in the opposite order through microscopic states identical to those of the "forward" solution, except that the magnetic field $\mathbb{B}$ has everywhere the opposite sign.[3]

Since the electric field $\mathbb{E}$ does not change sign between corresponding "forward" and "reverse" solutions, the Poynting vector ($\mathbb{E} \times \mathbb{B}$) does. Therefore, corresponding radiation fields propagate in opposite directions in systems evolving according to the two solutions. In particular, it is evident that the local spectral directional radiation which is emitted from matter at time $+t$ in a system evolving according to a particular "forward" solution corresponds with and is identical to the local spectral directional radiation which is absorbed by the matter at time $-t$ in a system evolving according to the corresponding "reverse" solution (and vice versa), except that the directions of propagation of these two radiation fields are reversed. We note here that, as may easily be seen from the definition of the Stokes vector, a field which is described by the Stokes vector $\mathbf{L} = (L_0, L_1, L_2, L_3)$

when it is regarded as propagating with respect to +t is described by the Stokes vector $\mathbb{L}^* = (L_0, L_1, -L_2, L_3)$ when it is regarded as propagating with respect to –t (i.e., when time is reversed).

We now consider two macroscopically identical enclosed cavities, each in thermodynamic equilibrium at temperature T. The cavities, by construction, differ only in that a time-independent magnetic field $-\mathbb{B}_0$ is externally applied to the first, while $+\mathbb{B}_0$ is applied to the second, and in that the first cavity is evolving microscopically according to a "forward" solution of the equations of motion, while the second is evolving according to the corresponding "reverse" solution.

Since the electromagnetic radiation being emitted in the first cavity at time +t must be strictly identical to the electromagnetic radiation being absorbed in the second cavity at time –t (again, except for a reversal of the direction of propagation), the amplitude of each polarization component of the local spectral directional electric field being emitted in the direction $\mathbf{\Omega}_r$ at wavelength λ_r in the first cavity at time +t must, individually, equal the amplitude of the corresponding polarization component of the local spectral directional electric field being absorbed from the direction $\mathbf{\Omega}_r$ at wavelength λ_r in the second cavity at time –t. Also, since the cavities are in thermodynamic equilibrium, all microscopic processes must, in fact, be stationary; hence, the time-averaged amplitude and the radiance (which is simply proportional to the square of this amplitude) associated with each polarization component of the local spectral directional electric field being emitted or absorbed in either cavity must be independent of time. We are therefore led to the following important relation concerning emission and absorption in the two cavities: The local spectral directional radiance associated with each polarization component of the electric field emitted in the first cavity must equal the local spectral directional radiance associated with the corresponding polarization component of the electric field absorbed in the second cavity.

Now, from the discussion at the end of Section 2 we know that the emitted local spectral directional radiance in the first cavity which is associated with the polarization component of the electric field described by any Stokes vector $\mathbf{P}$ of the form $(1, P_1, P_2, P_3)$, where $P_1^2 + P_2^2 + P_3^2 = 1$, is given by the quantity $(1/2)\mathbb{L}^e \cdot \mathbf{P}$. From Eq. (2) this may be rewritten as $(1/2)L_{bb}(\lambda_r, T)[\boldsymbol{\varepsilon}(\mathbf{R}, \mathbf{\Omega}_r, -\mathbb{B}_0, \lambda_r) \cdot \mathbf{P}]$, which for convenience we shall call E.

Likewise, from Section 2 and Eq. (3), we know that the absorbed local spectral directional radiance in the second cavity which is associated with the polarization component of the electric field described by the Stokes vector $\mathbb{P}^* = (1, P_1, -P_2, P_3)$ is given by the quantity $\boldsymbol{\alpha}(\mathbb{R}, \boldsymbol{\Omega}_r, +\mathbb{B}_o, \lambda_r) \cdot (1/2)[\mathbb{L}^i \cdot \mathbb{P}^*]\mathbb{P}^*$, which we shall call A. Furthermore, since the cavity is in equilibrium, we know that the radiation incident at each point on the walls of the cavity is blackbody radiation; viz., $\mathbb{L}^i = L_{bb}(1, 0, 0, 0)$; thus, A can be rewritten as $(1/2)L_{bb}(\lambda_r, T)[\boldsymbol{\alpha}(\mathbb{R}, \boldsymbol{\Omega}_r, +\mathbb{B}_o, \lambda_r) \cdot \mathbb{P}^*]$.

Since $\mathbb{P}^*$ is the time-reversed version of the Stokes vector $\mathbb{P}$, the reciprocity relation between emission and absorption in the two cavities requires A = E. As this equality must, in fact, hold for all choices of P_1, P_2, and P_3 for which $P_1^2 + P_2^2 + P_3^2 = 1$, we obtain finally at each point $\mathbb{R}$ on the walls of either cavity (for j = 0, 1, 2, 3):

$$\alpha_j(\boldsymbol{\Omega}_r, \mathbb{B}_o, \lambda_r) = (-1)^{\delta_{2j}} \varepsilon_j(\boldsymbol{\Omega}_r, -\mathbb{B}_o, \lambda_r) \quad , \tag{8}$$

which is known as Kirchhoff's law. We have obtained this law under the assumption that the radiation incident on the walls of the cavities is in actual equilibrium with the walls; however, as it is a relationship on the local material properties of the walls alone, it is evident that Eq. (8) will remain valid for arbitrary radiative environments so long as the material comprising the surface of the walls is itself in local thermodynamic equilibrium. By the same token, the validity of Eq. (8) extends to applied magnetic fields $\mathbb{B}_o$ which are time dependent but which vary sufficiently slowly that the radiating medium maintains a state of local thermodynamic equilibrium at each instant.[4]

We have discussed above the absorbed part of the incident electromagnetic field and how it corresponds to the emitted field in a macroscopically identical system that is evolving microscopically according to the "reverse" solution of the equations of motion. We now return to the two enclosed cavities and discuss the reflected part of the incident electromagnetic field. Because reflection is a process which inherently involves more than one direction and can involve more than one wavelength, we find that we must now keep track of photons, rather than the radiance alone.

We begin this time by considering a point on the wall inside the second cavity. As in the discussion above, since the cavity is in thermodynamic equilibrium, the radiation field incident on this point must be blackbody radiation. From Eq. (14), then, the rate $d\dot{N}_2$ (per unit surface area) at which photons associated with polarization state $\mathbb{I}$ in the small spectral band $d\lambda_i$ and from the small solid angle $d\mathbf{\Omega}_i$ are reflected into the small solid angle $d\mathbf{\Omega}_r$ with polarization state $\mathbf{O}$ in the small spectral band $d\lambda_r$ is given by:

$$d\dot{N}_2 = (1/4)[\mathbf{O} \cdot \boldsymbol{\rho}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, +\mathbf{B}_o, \lambda_r, \lambda_i) \cdot \mathbf{I}](\lambda_i/hc)L_{bb}(\lambda_i, T)\cos\beta_i d\mathbf{\Omega}_i d\lambda_i \cos\beta_r d\mathbf{\Omega}_r d\lambda_r \ .$$

Likewise, at the corresponding point in the first cavity, the rate $d\dot{N}_1$ (per unit surface area) at which photons associated with the polarization state $\mathbf{O}^*$ in the small spectral band $d\lambda_r$ and from the small solid angle $d\mathbf{\Omega}_r$ are reflected into the small solid angle $d\mathbf{\Omega}_i$ with polarization state $\mathbf{I}^*$ in the small spectral band $d\lambda_i$ is given by:

$$d\dot{N}_1 = (1/4)[\mathbf{I}^* \cdot \boldsymbol{\rho}(\mathbf{\Omega}_i, \mathbf{\Omega}_r, -\mathbf{B}_o, \lambda_i, \lambda_r) \cdot \mathbf{O}^*](\lambda_r/hc)L_{bb}(\lambda_r, T)\cos\beta_r d\mathbf{\Omega}_r d\lambda_r \cos\beta_i d\mathbf{\Omega}_i d\lambda_i \ .$$

From our previous discussion concerning corresponding radiation fields in systems evolving microscopically according to corresponding "forward" and "reverse" solutions to the equations of motion, it should be evident that $d\dot{N}_1$ and $d\dot{N}_2$ in fact represent the photon flux rates (associated with any two polarization components of the fields involved in the reflection) of two such corresponding fields. Therefore, $d\dot{N}_1$ and $d\dot{N}_2$ must be equal. Since the equality must hold for any choice of polarization components, we must have, finally, at each point on the walls of either cavity (for j, k = 0, 1, 2, 3):

$$\lambda_i L_{bb}(\lambda_i, T)\rho_{jk}(\mathbf{\Omega}_r, \mathbf{\Omega}_i, \mathbf{B}_o, \lambda_r, \lambda_i) = (-1)^{\delta_{2j} + \delta_{2k}} \lambda_r L_{bb}(\lambda_r, T)\rho_{kj}(\mathbf{\Omega}_i, \mathbf{\Omega}_r, -\mathbf{B}_o, \lambda_i, \lambda_r) . \quad (7)$$

Equation (7) expresses the generalized reciprocity relations on $\boldsymbol{\rho}$. Restrictions on the validity of this result are the same as those discussed above in connection with Kirchhoff's law.

REFERENCES AND NOTES, APPENDIX A

1. R. Siegel and J.R. Howell, *Thermal Radiation Heat Transfer*, 2nd ed., McGraw-Hill, New York, 1981.

2. H.C. van de Hulst, *Multiple Light Scattering*, Vol. 1, Academic Press, New York, 1980, Chapter 3. See also p. 173 of Ref. 1.

3. Systems whose microscopic evolution is independent of the sign of the applied magnetic field are termed "reciprocal". This term is also used, however, to describe systems whose macroscopic properties of interest are independent of the sign of the applied magnetic field (even though aspects of the microscopic evolution may not be).

4. A point perhaps worthy of some elaboration concerns the distinction between the applied magnetic field $\mathbb{B}_0$ and the magnetic field associated with the incident electromagnetic radiation. The distinction between these two fields is, in fact, solely a matter of separation of time scales: We are interested in describing the radiative behavior of material media on a time scale which is long compared with the wavelength period of the "incident" radiation, but which is short compared with the characteristic time for variations in $\mathbb{B}_0$. Indeed, the Stokes vector formalism, as we have employed it in this article, is inherently intended to describe the outcome of measurements which integrate over many oscillations of the incident electromagnetic field, but which sample the applied field instantaneously.

APPENDIX B
AN EXAMPLE "NONRECIPROCAL" SYSTEM

As a simple and practical example of a "nonreciprocal" system we discuss a device known as a radiation isolator. This device is transparent to radiation incident from one direction and opaque to radiation incident from the opposite direction--a true one-way filter--and is useful in preventing feedback between elements in an optical system.

We consider here one type of isolator that consists of three elements in series: a linear polarizer; a Faraday rotator adjusted to rotate the polarization plane of an incident linearly polarized beam by +45 deg; and a second linear polarizer whose transmission plane is rotated by +45 deg with respect to the first polarizer. Both polarizers absorb (rather than reflect) the radiation they do not pass.

The key element of the isolator is the Faraday rotator, which has a magnetic field $\mathbf{B}_0$ externally applied along the propagation axis and is the actual "nonreciprocal" component. (A wide variety of solids, liquids, and gases exhibit the Faraday effect, but glasses are ordinarily employed as the media of choice as they typically produce the largest rotation angle for a given field strength.) A Faraday rotator is a nonabsorbing device which causes linearly polarized radiation incident from one direction to be rotated in a right-handed sense, and linearly polarized radiation incident from the opposite direction to be rotated by an equal amount in a left-handed sense. Reversing the direction of $\mathbf{B}_0$ causes the senses of rotation to be exchanged.

We confine our attention to the radiative properties of the isolator as viewed from one side along the normal propagation axis. The coordinate system for defining the Stokes vector can be arranged so that incident radiation with Stokes vector (1, 0, 1, 0) passes through the isolator without attenuation, whereas incident radiation with Stokes vector (1, 0, −1, 0) is entirely absorbed by the first polarizer. The absorptivity vector for the isolator is thus simply that of the first polarizer:

$$\alpha(\mathbf{B}_0) = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{pmatrix} .$$

The emission from the isolator, on the other hand, is completely unpolarized, as can be readily seen. The emissivity of the first polarizer alone is (1/2, 0, 1/2, 0). The second polarizer, by itself, would have emissivity (1/2, −1/2, 0, 0); however, after passage through the Faraday rotator, the plane of polarization of the emitted radiation from the second polarizer is rotated and passes without attenuation through the first polarizer. Hence, the emissivity vector contributed by the second polarizer to the isolator is (1/2, 0, −1/2, 0). The full emissivity vector of the isolator is, in this case, simply the sum of the contributions of the two polarizers, and is therefore:

$$\boldsymbol{\varepsilon}(\mathbb{B}_o) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

We have thus obtained the interesting result that along the propagation axis the isolator emits unpolarized radiation, exactly like a blackbody, even though it is completely transparent to radiation with Stokes vector (1, 0, 1, 0).

If the direction of the magnetic field applied to the Faraday rotator is reversed, the plane of polarization of the emitted radiation from the second polarizer is rotated in the opposite sense and is blocked by the first polarizer. The second polarizer thus makes no contribution to the emissivity vector of the isolator, and the emissivity of the isolator is simply that of the first polarizer:

$$\boldsymbol{\varepsilon}(-\mathbb{B}_o) = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} .$$

Summarizing, we have obtained for the isolator

$$\alpha_j(\mathbb{B}_o) = (-1)^{\delta_{2j}} \varepsilon_j(-\mathbb{B}_o) \neq \varepsilon_j(\mathbb{B}_o) ,$$

which is in agreement with Eq. (8).

APPENDIX C
DISCUSSION OF ASSUMPTIONS

In obtaining the results of this article we made three main assumptions, which are listed here:

1. We assumed the surface of the opaque body to be in local thermodynamic equilibrium. This assumption enables us to define a temperature at each position $\mathbf{R}$ (on or near the surface) and is valid in all but the most extreme of radiative environments.[1]
2. We assumed multiphoton processes to be negligible. This assumption was made in writing Eqs. (1) and (3), and insures that reflection and absorption are processes linear in the Stokes vector of the incident radiation. As is the case for our first assumption, this assumption is also valid except when the incident electromagnetic fields are enormously strong.
3. Finally, we assumed the existence of a certain length scale $\mathbf{l}$ for the opaque body. In particular, we assumed it to be conceptually possible to subdivide the surface into elements with sides of length $\mathbf{l}$ which are, at the wavelengths of interest, (a) sufficiently small that: (1) each element is effectively planar, (2) each element has a temperature which is effectively uniform across it (and throughout a thickness below it large compared with the penetration depth of the incident radiation), and (3) the incident radiation at any given angle of incidence is effectively uniform across each element; and (b) sufficiently large that effectively all externally incident radiation is either absorbed or reflected back out of the same element into the hemisphere above. (We note that the existence of $\mathbf{l}$ may, in fact, be regarded as a technical criterion for opacity, rather than an additional assumption.)

REFERENCE, APPENDIX C

1. R. Siegel and J.R. Howell, *Thermal Radiation Heat Transfer*, 2nd ed., McGraw-Hill, New York, 1981, pp. 58 and 446-447.